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Exercises for 'Topics in complex analysis'

(12/11/2025)

H 9.1 (A general version of Bloch's theorem)
Prove that if $f : U \to \mathbb{C}$ is holomorphic and $f'(c) \neq 0$ at a point $c \in U$, then $f(U)$ contains a ball of radius $(\frac{3}{2} - \sqrt{2})s|f'(c)|$, for every $0 < s < \operatorname{dist}(c, \partial U)$. In particular, show that if $f : \mathbb{C} \to \mathbb{C}$ is entire and non-constant, then $f(\mathbb{C})$ contains balls of arbitrarily large radius.

H 9.2 (Biholomorphic functions from $B_1(0)$ to $B_1(0)$)
For $z_0 \in B_1(0)$ define the function $\varphi_{z_0} : B_1(0) \to \mathbb{C}$, called a Blaschke factor, by

$$\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

Show that φ_{z_0} is a biholomorphic map from $B_1(0)$ to $B_1(0)$. Moreover, prove that any biholomorphic map $f : B_1(0) \to B_1(0)$ is of the form $f(z) = a\varphi_{z_0}(z)$ for some $z_0 \in B_1(0)$ and $a \in \partial B_1(0)$.

Hint: For the second part, recall the Schwarz lemma (Lemma 7.2 in the notes).

H 9.3 (Improving the constant in Bloch's theorem)
The purpose of this exercise is to improve the constant $\left(\frac{3}{2} - \sqrt{2}\right)$ appearing in Bloch's theorem. We will show that for every $f \in \mathcal{H}(\overline{B_1(0)})$ with $f'(0) = 1$, the image $f(B_1(0))$ contains a disc of radius $\frac{3}{2}\sqrt{2} - 2$.

a) Motivated by H 9.1, we look for a function $F \in \mathcal{H}(\overline{B_1(0)})$ such that $f(B_1(0)) = F(B_1(0))$ with maximal value $|F'(0)|$. To make this precise, denote by

$$\mathcal{F} := \{f \circ (a\varphi_{z_0}) : z_0 \in B_1(0),\ a \in \partial B_1(0)\},$$

where $\varphi_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z}$ as in H 9.2. Show that if $h = f \circ (a\varphi_{z_0}) \in \mathcal{F}$ then $h \in \mathcal{H}(\overline{B_1(0)})$, $h(B_1(0)) = f(B_1(0))$, and $|h'(0)| = |f'(-az_0)|(1 - |z_0|^2)$.

b) Denote by q the maximizer of the map $\overline{B_1(0)} \ni z \mapsto |f'(z)|(1 - |z|^2)$. Show that $q \in B_1(0)$ and that denoting $F = f \circ \varphi_{-q}$ it holds that

$$|F'(z)| \leq \frac{|f'(q)|(1 - |q|^2)}{1 - |z|^2}.$$

c) Deduce that $|F'(z)| \leq 2|F'(0)|$ for all $|z| \leq \frac{1}{2}\sqrt{2}$. Conclude the proof using Step 2 of the proof of Bloch's theorem in the lecture notes.

Remark: It is an open problem to find the largest possible radius (say B) allowed in Bloch's theorem. The best known upper and lower bounds are

$$0.4332127... = \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} \leq B \leq \frac{1}{\sqrt{1 + \sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} = 0.4718617...,$$

and it is conjectured that B is equal to the upper bound.

H 9.4 (Optional and difficult)
Let $G \subset \mathbb{C}$ be a simply-connected domain and let $f : G \to \mathbb{C}$ be holomorphic. Set

$$P = \{z \in G : \ f(z) \in \{\pm 1\}\}.$$

Show that there exists a holomorphic function $g : G \to \mathbb{C}$ such that $f = \cos(g)$ if and only if for each $z_0 \in P$ the function $z \mapsto f(z) - f(z_0)$ has a zero of even order at $z = z_0$.

Hint: Show that one can define the function $g(z) = -i \log\left(f(z) + \sqrt{f(z)+1}\sqrt{f(z)-1}\right)$. To define the square-root, use the Weierstrass product theorem.

H 9.5 (First applications of Picard's little theorem)
a) Show that every meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that omits three distinct values $a, b, c \in \mathbb{C}$ is constant.

b) Give an example of a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ that omits the two values $0, 1 \in \mathbb{C}$.

c) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that $f \circ f$ has a fixed point, except when $f(z) = z + b$ for some $b \in \mathbb{C} \setminus \{0\}$.

Hint: Consider the function $z \mapsto \frac{f(f(z))-z}{f(z)-z}$. Show that it is equal to a constant c and differentiate it. Then deduce that $f' \circ f$ omits 0 and c. Finally, show that this implies that f' is constant.